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# On the Geometry of Cake Division

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We study partitions of a "cake" $C$ among $n$ players. Each player uses a countably additive non-atomic probability measure to evaluate the sizes of pieces of cake. If the players' measures are $m_1, m_2, \dots, m_n$, then the "Individual Pieces Set," which we studied before (2000, *J. Math. Econom.* **33**, 401–424), is the set $\{(m_1(P_1), m_2(P_2), \dots, m_n(P_n)) \colon \langle P_1, P_2, \dots, P_n \rangle$ is a partition of $C\}$. We continue our study of this set here. Our motivating question is: What are the possible shapes of such sets? We give an exact characterization for $n = 2$, establish some partial results for $n = 3$, and close with open questions. 

*Key Words:* fair division; Pareto optimal.

## 0. INTRODUCTION

We consider the cake division problem. Our "cake" is a set $C$. The cake is to be partitioned among $n$ players, whom we shall refer to as Player 1, Player 2, ..., Player $n$. Each player has a countably additive, non-atomic probability measure defined on some $\sigma$-algebra of subsets of $C$ that is used to evaluate the size of pieces of cake (i.e., subsets of $C$). Throughout the paper, a "measure" shall always mean a countably additive, non-atomic, probability measure. Whenever a subset $X$ of $C$ is mentioned, we assume that $X$ is a member of some common $\sigma$-algebra on which all of our measures are defined.

Various notions of what it means for a partition of the cake among $n$ players to be a "good" partition have been considered. See, for example, Barbanel [1] or Dubins and Spanier [6]. Our concern in this paper is not directly with these notions (although we shall find it useful to consider one such notion, Pareto optimality). Rather, we shall be concerned with certain geometric properties of the set of all possible partitions.

Any cake $C$ and collection of $n$ measures on $C$ naturally give rise to two sets. These are the Radon–Nikodym Set and the Individual Pieces Set,



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both of which we studied in [3]. In this paper, we continue our study of the Individual Pieces Set, which we shall denote by IPS. Our overall focus is: What sorts of IPSs are possible? In other words, given a set $A$ satisfying certain reasonable conditions, can we find a cake $C$ and corresponding measures so that $A$ is the IPS corresponding to this cake and these measures? In Section 2, we give an affirmative answer for $n = 2$. In Section 3, we consider the case of $n = 3$. We give various partial results and counterexamples and close with open questions.

## 1. DEFINITIONS, BACKGROUND RESULTS, AND BASIC FACTS ABOUT THE IPS

For this section, let $C$ be a cake (i.e., a set) and let $m_1, m_2, \ldots, m_n$ be measures on $C$. For any partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of $C$, let $m(P) = (m_1(P_1), m_2(P_2), \ldots, m_n(P_n))$.

The Individual Pieces Set associated with $C$ and the measures $m_1, m_2, \ldots, m_n$ is $\{m(P) : P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. We shall denote this set by IPS($C$; $m_1, m_2, \ldots, m_n$), or simply by IPS if $C$ and $m_1, m_2, \ldots, m_n$ are clear by context. Note that IPS($C; m_1, m_2, \ldots, m_n) \subseteq \mathbb{R}^n$.

It will be convenient to view the IPS as arising in a natural way from what we shall call the Full Individual Pieces Set, or FIPS. For any partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of $C$, let $m_{\mathrm{F}}(P)$ be the $n \times n$ matrix $[m_i(P_j)]_{i, j \leq n}$. The Full Individual Pieces Set, or FIPS, is the set $\{m_{\mathrm{F}}(P) : P = \langle P_1, P_2, \ldots, P_n \rangle$ is a partition of $C\}$. It follows from the theorem of Dvoretsky, Wald, and Wolfovitz (see [6] or [7]) that the FIPS is closed and convex, since the measures are atomless. The IPS is simply the set of diagonals of the elements of the FIPS, and thus the IPS is closed and convex.

If, instead of considering the set of diagonals of elements of the FIPS, we consider the set of first columns (or the set of all $i$th columns for any fixed $i \leq n$) of the FIPS, then we obtain $\{(m_1(A), m_2(A), \ldots, m_n(A)) : A \subseteq C\}$. This set is called a zonoid. It has been studied by Bolker [5] and Neyman [9].

We wish to understand the shape of the IPS. In this section, we discuss the general case of $n$ players. In the next two sections, we shall consider $n = 2$ and $n = 3$ and thus will be able to make use of our geometric intuition in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We first note that if $P = \langle \mathrm{C}, \emptyset, \emptyset, \ldots, \emptyset \rangle$ then $m(P) = (1, 0, 0, \ldots, 0) \in$ IPS. Similarly, $(0, 1, 0, 0, \ldots, 0, 0), (0, 0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, 0, \ldots, 0, 1) \in$ IPS. By convexity, it follows that all convex combinations of these elements of the IPS are in the IPS. In other words, the $(n - 1)$-simplex is a subset of the IPS. It is not hard to see that the IPS consists precisely of the

$(n-1)$-simplex if and only if the measures $m_1, m_2, \ldots, m_n$ are identical. Also, since our measures take values in the closed interval $[0,1]$, the IPS must be a subset of the closed unit hypercube $[0,1]^n$ (i.e., the $n$-fold Cartesian product of the closed unit interval $[0,1]$ with itself).

It will be useful for us to discuss the "outer and inner boundaries" of the IPS. Since the IPS is convex, its boundary is a connected surface in $\mathbb{R}^n$. Also, by the preceding paragraph, the points $(1,0,0,\ldots,0), (0,1,0,0,\ldots,0,0), (0,0,1,0,\ldots,0,0), \ldots, (0,0,0,0,\ldots,0,1)$ are on the boundary. The outer boundary consists of all points $(x_1, x_2, \ldots, x_n)$ of the boundary with $x_1 + x_2 + \cdots + x_n \geq 1$, and the inner boundary consists of all points $(x_1, x_2, \ldots, x_n)$ of the boundary with $x_1 + x_2 + \cdots + x_n \leq 1$. Equivalently, the outer boundary of the IPS is the subset of the boundary that is not on the side of the simplex closest to the origin, and the inner boundary of the IPS is the subset of the boundary that is not on the side of the simplex farthest from the origin. Note that the outer and inner boundaries each include the points $(1,0,0,\ldots,0), (0,1,0,0,\ldots,0,0), (0,0,1,0,\ldots,0,0), \ldots,$

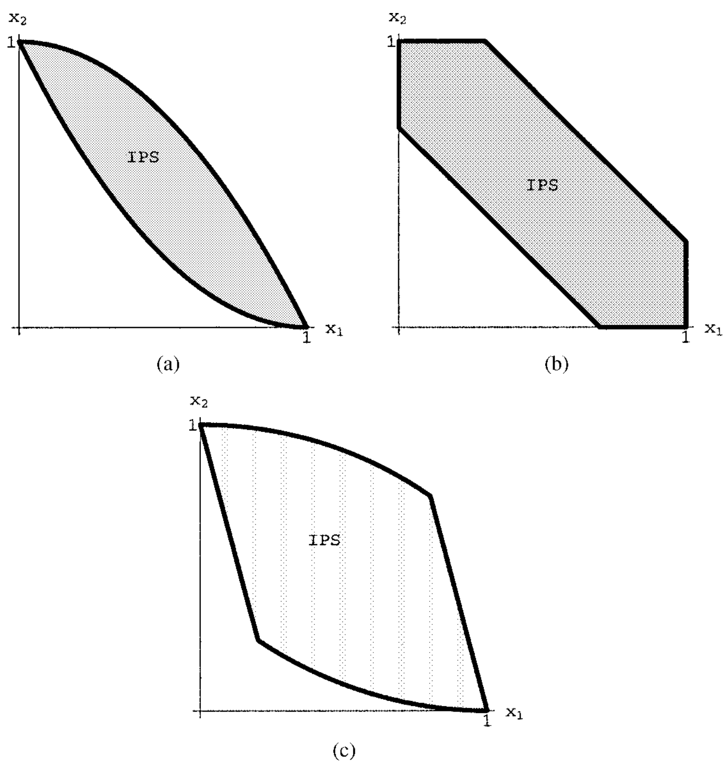


FIGURE 1

$(0, 0, 0, 0, \ldots, 0, 1)$. Figure 1 shows three IPSs for $n = 2$, with outer and inner boundaries darkened.

Theorem 3 of the next section will tell us that these figures really are possible IPSs. In other words, for each of these three figures, there is a cake $C$ and corresponding measures $m_1$ and $m_2$ such that the given figure is IPS$(C; m_1, m_2)$. In [3], we studied the underlying reasons why IPS boundaries may consist of all straight lines (as in Fig. 1b) or all (non-straight-line) curves (as in Fig. 1a) or sometimes a mixture of the two (as in Fig. 1c).

A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of $C$ is Pareto maximal if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ do we have $m_i(Q_i) \geq m_i(P_i)$ for each $i = 1, 2, \ldots, n$, with at least one inequality strict. A partition $P = \langle P_1, P_2, \ldots, P_n \rangle$ of $C$ is Pareto minimal if and only if for no partition $Q = \langle Q_1, Q_2, \ldots, Q_n \rangle$ do we have $m_i(Q_i) \leq m_i(P_i)$ for each $i = 1, 2, \ldots, n$, with at least one inequality strict. (The term "Pareto optimal" is more commonly used instead of our "Pareto maximal." We find "Pareto maximal" and "Pareto minimal" to be more appropriate terms, since "optimal" can sometimes mean "maximal" and sometimes "minimal," depending on the context.)

It is not hard to see that if a partition $P$ is Pareto maximal then $m(P)$ is on the outer boundary of the IPS, and if $P$ is Pareto minimal then $m(P)$ is on the inner boundary of the IPS. The converse of each of these statements is true if and only if our measures are absolutely continuous with respect to each other (i.e., if and only if whenever a subset of $C$ is of measure zero with respect to one measure, then it is of measure zero with respect to all measures).

To understand this last statement, let us consider the $n = 2$ context. Suppose that $X$ is a piece of cake, $m_1(X) > 0$, and $m_2(X) = 0$. Clearly, any Pareto maximal partition of $C$ must give all of $X$ to Player 1. An example of this situation is illustrated by the IPS of Fig. 1b. In the figure, the effect of this set $X$ is seen in the existence of a horizontal line segment on the outer boundary, connecting to the point $(0, 1)$. No point on this segment corresponds to a Pareto maximal partition, except for the right endpoint of the segment. This example illustrates the following three facts for the $n = 2$ case:

i. The existence of a horizontal line segment or a vertical line segment on the outer boundary of the IPS must occur in the upper left or the lower right of the outer boundary, respectively (i.e., one endpoint must be $(0, 1)$ or $(1, 0)$, respectively).

ii. Such a horizontal or vertical line segment corresponds to the failure of absolute continuity of the associated measures.

iii. A partition $P$ is Pareto maximal if and only if $m(P)$ is on the outer boundary of the IPS and is either not on such a horizontal or vertical line segment, or is at the endpoint of the segment that is not $(0,1)$ or $(1,0)$.

Analogous facts are true for Pareto minimality (as illustrated by the inner boundary of Fig. 1b) and for $n > 2$.

In [4], we used the notion of an outer boundary to categorize Pareto maximality in terms of the maximization of convex combinations of measures. An analogous argument would establish a categorization of Pareto minimality in terms of the minimization of convex combinations of measures.

In [2], we studied the notion of partition ratios. Suppose $P = \langle P_1, P_2, \dots, P_n \rangle$ is a partition of $C$ and assume that $m_i(P_i) \neq 0$ for each $i$. In other words, we assume that each player believes that he or she has a piece of cake of positive measure.

The definitions from that paper that we will need here are the following:

DEFINITIONS. i. For each $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, the $i - j$ maximality partition ratio, denoted $\mathrm{pr}_{ij}$, is given by $\mathrm{pr}_{ij} = \sup\{m_j(X)/m_i(X) \colon X \subseteq P_i,\ m_i(X) \neq 0\}$.

ii. A sequence of the form $\langle \mathrm{pr}_{i_1 i_2}, \mathrm{pr}_{i_2 i_3}, \dots, \mathrm{pr}_{i_{k-1} i_k}, \mathrm{pr}_{i_k i_1} \rangle$, where the $i_j$ are distinct, is called a maximality cyclic sequence.

iii. For any maximality cyclic sequence $\varphi = \langle \mathrm{pr}_{i_1 i_2}, \mathrm{pr}_{i_2 i_3}, \dots, \mathrm{pr}_{i_{k-1} i_k}, \mathrm{pr}_{i_k i_1} \rangle$ the maximality cyclic product of $\varphi$, denoted by $\mathrm{CP}(\varphi)$, is the product $\mathrm{pr}_{i_1 i_2} \mathrm{pr}_{i_2 i_3}, \dots, \mathrm{pr}_{i_{k-1} i_k} \mathrm{pr}_{i_k i_1}$.

In [2], the word "maximality" was not used in these definitions. We have inserted it here because we consider "minimality partition ratios" below. Of course, all of the definitions above depend on the partition $P$.

We note that it is possible that for some $i, j$, $\mathrm{pr}_{ij} = \infty$.

The intuition behind these maximality partition ratios is that they tell us about the relative weights that two players put on the piece of cake given to one of the two players. For example, if $\mathrm{pr}_{ij}$ is near 0, then Player $j$ puts a small value on all of $P_i$ compared with Player $i$'s evaluation, whereas if $\mathrm{pr}_{ij}$ is very large, then there are parts of $P_i$ that Player $j$ values much more than does Player $i$.

The main result involving these notions is Theorem 1. We proved the theorem in [2], using ideas due to Weller (see [10]). For this theorem we adopt the following conventions:

i. for any positive $k$, $(\infty)(k) > 1$, *and*

ii. $(\infty)(0) = 0$.

THEOREM 1. *$P$ is Pareto maximal if and only if for any maximality cyclic sequence $\varphi$, $CP(\varphi) \leq 1$.*

Although we did not discuss partition ratios for Pareto minimality in [2], the appropriate definitions, theorem, and proof are entirely analogous to those above. We shall need this material in what follows.

DEFINITION. For each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the $i - j$ minimality partition ratio is given by $\mathrm{qr}_{ij} = \inf\{m_j(X)/m_i(X) : X \subseteq P_i, m_i(X) \neq 0\}$.

The definitions of minimality cyclic sequence and minimality cyclic product are precisely as in the maximality case.

THEOREM 2. *P is Pareto minimal if and only if for any minimality cyclic sequence $\varphi$, $CP(\varphi) \geq 1$.*

## 2. THE $n = 2$ CASE

For this section, we assume that there are two players. Thus, for any cake $C$ and associated measures $m_1$ and $m_2$, $\mathrm{IPS}(C; m_1, m_2) = \{(m_1(P_1), m_2(P_2)) : \langle P_1, P_2 \rangle$ is a partition of $C\} \subseteq \mathbb{R}^2$. Then the 1-simplex, which is the line segment between $(1, 0)$ and $(0, 1)$, is in $\mathrm{IPS}(C; m_1, m_2)$. We shall refer to our horizontal and vertical axes as the $x_1$ axis and $x_2$ axis, respectively.

In the case of two players, the IPS possesses a nice symmetry property.

LEMMA 1. *For any cake $C$, and associated measures $m_1$ and $m_2$, $\mathrm{IPS}(C; m_1, m_2)$ is centrally symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.*

*Proof of Lemma.* Suppose $C$ is a cake, $m_1$ and $m_2$ are measures on $C$, and $(a, b) \in \mathrm{IPS}(C; m_1, m_2)$. Then, for some partition $\langle P_1, P_2 \rangle$ of $C$, $(a, b) = (m_1(P_1), m_2(P_2))$. But then $(m_1(P_2), m_2(P_1)) = (1 - m_1(P_1), 1 - m_2(P_2)) = (1 - a, 1 - b) \in \mathrm{IPS}(C; m_1, m_2)$. Since $(1 - a, 1 - b)$ is the reflection of $(a, b)$ about the point $(\frac{1}{2}, \frac{1}{2})$, the lemma follows. ∎

This symmetry is evident in the examples shown in Fig. 1.

We will consider symmetry of the IPS for $n > 2$ in the next section. For now, we simply note that there is no obvious generalization of Lemma 1 and its proof to $n > 2$.

Lemma 1, together with our discussion in the previous section, begins to give us a picture of what the IPS looks like. We wish to consider the question: What are the possible shapes of the IPS? Our discussion so far tells us that the IPS must

i. be a subset of the closed unit square $[0, 1] \times [0, 1]$;
ii. contain the points $(1, 0)$ and $(0, 1)$;
iii. be closed;

iv. be convex;

v. be centrally symmetric about the point $(\frac{1}{2}, \frac{1}{2})$.

The main result of this section is that these conditions completely characterize the possible shapes of the IPS. In particular, we have

THEOREM 3. *Let $A$ be a subset of $\mathbb{R}^2$. There exists a cake $C$ and measures $m_1$ and $m_2$ on $C$ such that $A = \mathrm{IPS}(C; m_1, m_2)$ if and only if $A$ satisfies the following five conditions*:

a. $A \subseteq [0,1] \times [0,1]$;

b. $(1,0), (0,1) \in A$;

c. *$A$ is closed*;

d. *$A$ is convex*;

e. *$A$ is centrally symmetric about $(\frac{1}{2}, \frac{1}{2})$.*

We wish to prove two simple lemmas that will be useful in the proof of the theorem. For convenience, let $\mathrm{IPS}^{\mathrm{out}}$ denote the outer boundary of the IPS.

Suppose that $U$ is any set that satisfies the five conditions of the theorem. Conditions a and d tell us that the boundary of $U$ is a connected and closed curve. Conditions a and b imply that this curve contains the points $(1,0)$ and $(0,1)$. We can define the outer boundary of $U$, denoted by $U^{\mathrm{out}}$, in precisely the same way that we defined the outer boundary of the IPS in the previous section. Thus, $U^{\mathrm{out}}$ consists of all points $(x_1, x_2)$ on the boundary of $U$ for which $x_1 + x_2 \geq 1$. Equivalently, $U^{\mathrm{out}}$ is the subset of the boundary of $U$ that includes the points $(1,0)$ and $(0,1)$ and all other points of the boundary that are not on the same side of the segment connecting $(1,0)$ and $(0,1)$, as is the origin.

LEMMA 2. *Suppose that $U$ and $V$ each satisfy the five conditions of the theorem. If $U^{\mathrm{out}} \subseteq V^{\mathrm{out}}$, then $U^{\mathrm{out}} = V^{\mathrm{out}}$.*

*Proof of Lemma.* Assume that $U$ and $V$ each satisfy the five conditions of the lemma. By condition d, $U^{\mathrm{out}}$ and $V^{\mathrm{out}}$ are each connected curves. By conditions a and b, each has endpoints $(1,0)$ and $(0,1)$. Then, the only way we could have $U^{\mathrm{out}} \subseteq V^{\mathrm{out}}$ and $U^{\mathrm{out}} \neq V^{\mathrm{out}}$ is if $V^{\mathrm{out}}$ is self-intersecting, and this is certainly not possible, given that $V^{\mathrm{out}}$ is a subset of the boundary of a convex set. ∎

A more explicit proof of Lemma 2 could be constructed by using the fact that $U^{\mathrm{out}}$ and $V^{\mathrm{out}}$ can each be viewed as the range of a one-to-one function of the angle between the positive $x_1$ axis and the line segments connecting the origin to points on the curve. The two functions have the same domain ($[0, 90°]$). It is then straightforward to show that $U^{\mathrm{out}} \subseteq V^{\mathrm{out}}$ implies $U^{\mathrm{out}} = V^{\mathrm{out}}$.

LEMMA 3. *Any set satisfying the five conditions of the theorem is uniquely determined by its outer boundary. In other words, if $U$ and $V$ each satisfy the five conditions of the theorem and $U^{\text{out}} = V^{\text{out}}$, then $U = V$.*

*Proof of Lemma.* Let $U$ be any set satisfying the five conditions of the theorem. Define the inner boundary of $U$ in the obvious way. By condition e, the outer boundary uniquely determines the inner boundary. By conditions a and b the outer boundary has endpoints $(1,0)$, $(0,1)$. Hence, the inner boundary has endpoints $(1,0)$ and $(0,1)$. It follows that the outer and inner boundaries together make a closed curved, and, by conditions c and d, $U$ consists of this curve together with the region enclosed by this curve. This determination of $U$ from its outer boundary is clearly unique. ∎

*Proof of Theorem* 3. The forward direction follows from our previous discussion.

For the reverse direction, suppose $A$ is a subset of $\mathbb{R}^2$ that satisfies the five given conditions. We must find a cake $C$ and measures $m_1$ and $m_2$ on $C$ such that $A = \text{IPS}(C; m_1, m_2)$.

We define our cake $C$ to be $A^{\text{out}}$. For any $B \subseteq C$, let $B_1$ be the projection of $B$ onto the $x_1$ axis and let $B_2$ be the projection of $B$ onto the $x_2$ axis. Let $m_{\text{Leb}}$ be Lebesgue measure on the real line. Finally, we define $m_1$ and $m_2$ on $C$ as follows: for any such $B \subseteq C$, $m_1(B) = m_{\text{Leb}}(B_1)$ and $m_2(B) = m_{\text{Leb}}(B_2)$. It is easy to see that $m_1$ and $m_2$ are (countably additive, non-atomic, probability) measures on $C$.

We claim that $A = \text{IPS}(C; m_1, m_2)$. For simplicity, let IPS denote $\text{IPS}(C; m_1, m_2)$ for the remainder of the proof. Since we already know that the forward direction of the theorem is true, we know that IPS satisfies the five conditions of the theorem. Thus, since both $A$ and the IPS satisfy these conditions, it follows from Lemmas 2 and 3 that to show $A = \text{IPS}$, it suffices to show $A^{\text{out}} \subseteq \text{IPS}^{\text{out}}$.

We need to develop some notation. For any $(p, q) \in C$, let $\text{UL}(p, q)$ be that portion of the curve $C$ that is between $(0,1)$ and $(p, q)$, including $(0,1)$ but not $(p, q)$, and let $\text{LR}(p, q)$ be that portion of the curve $C$ that is between $(p, q)$ and $(1,0)$, including both $(p, q)$ and $(1,0)$. (Our specification of which set includes the point $(p, q)$ is arbitrary, since any point will have measure 0.) UL and LR are meant to denote "upper left" and "lower right."

Suppose that $(p, q) \in A^{\text{out}}$ and hence $(p, q) \in C$. We wish to show that $(p, q) \in \text{IPS}^{\text{out}}$. Consider the partition of $C$ given by $\langle \text{UL}(p, q), \text{LR}(p, q) \rangle$. We have

$$\begin{aligned} m_1(\text{UL}(p, q)) &= m_{\text{Leb}}(\text{Projection of UL}(p, q) \text{ onto the } x_1 \text{ axis}) \\ &= m_{\text{Leb}}([0, p)) = p \end{aligned}$$

and

$$\begin{aligned} m_2(\mathrm{LR}(p,q)) &= m_{\mathrm{Leb}}(\text{Projection of } \mathrm{LR}(p,q) \text{ onto the } x_2 \text{ axis}) \\ &= m_{\mathrm{Leb}}([0,q]) = q. \end{aligned}$$

This tells us that $(p,q) \in \mathrm{IPS}$. We need to show that $(p,q) \in \mathrm{IPS}^{\mathrm{out}}$. We consider three cases:

*Case* 1. $p = 1$. In this case, $(p,q)$ is at or above the point $(1,0)$. Then, since we know that $\mathrm{IPS} \subseteq [0,1] \times [0,1]$, it follows that $(p,q) \in \mathrm{IPS}^{\mathrm{out}}$.

*Case* 2. $q = 1$. In this case, $(p,q)$ is at or to the right of the point $(0,1)$. Then, as above, since we know that $\mathrm{IPS} \subseteq [0,1] \times [0,1]$, it follows that $(p,q) \in \mathrm{IPS}^{\mathrm{out}}$.

*Case* 3. $p \neq 1$ and $q \neq 1$. Then, as discussed in the previous section, it suffices to show that $\langle \mathrm{UL}(p,q), \mathrm{LR}(p,q) \rangle$ is a Pareto maximal partition of $C$. We establish this by using maximality partition ratios.

We must show that for all maximality cyclic sequences $\varphi$, $\mathrm{CP}(\varphi) \leq 1$. We need only consider the product of the two maximality partition ratios that exist in this $n = 2$ case. We have

$$\mathrm{pr}_{12} = \sup\left\{ \frac{m_2(X)}{m_1(X)} : X \subseteq \mathrm{UL}(\langle p,q \rangle),\, m_1(X) \neq 0 \right\} \qquad \text{and}$$

$$\mathrm{pr}_{21} = \sup\left\{ \frac{m_1(X)}{m_2(X)} : X \subseteq \mathrm{LR}(\langle p,q \rangle),\, m_2(X) \neq 0 \right\}.$$

We note that since we are in Case 3, $m_1(\mathrm{UL}(p,q)) \neq 0$ and $m_2(\mathrm{LR}(p,q)) \neq 0$. We wish to show that $\mathrm{pr}_{12}\,\mathrm{pr}_{21} \leq 1$. This is implied by the following claim:

*Claim*. For any $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq \mathrm{LR}(p,q)$, $m_2(X)/m_1(X) \leq m_2(Y)/m_1(Y)$.

*Proof of Claim*. For any line $\ell$, let $m(\ell)$ be the slope of $\ell$. We begin by noting that the conditions of the theorem easily imply that, given two secant lines $\ell_1$ and $\ell_2$ to $C$, if $\ell_1$ connects two points of $\mathrm{UL}(p,q)$ and $\ell_2$ connects two points of $\mathrm{LR}(p,q)$, then $m(\ell_1) \geq m(\ell_2)$. Also, since both of these numbers are negative, $|m(\ell_1)| \leq |m(\ell_2)|$.

Let $X$ be any connected piece of $C$. In particular, let us assume that $X$ runs between the two points $(a,b)$ and $(c,d)$ of $C$. Then, $m_1(X) = |c-a|$ and $m_2(X) = |d-b|$, and so $m_2(X)/m_1(X) = |(d-b)/(c-a)|$, which is the absolute value of the slope of the secant line to $C$ between $(a,b)$ and $(c,d)$. Then, by the previous paragraph, if $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq \mathrm{LR}(p,q)$

are each connected pieces of $C$, then $m_2(X)/m_1(X) \leq m_2(Y)/m_1(Y)$. This establishes the claim for the special case where $X$ and $Y$ are each connected pieces of $C$. Then, it is straightforward to verify that the claim holds if $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq \mathrm{LR}(p,q)$ are each (finite or infinite) unions of connected pieces of $C$.

We shall say that a piece of $C$ is open if and only if its projection to both the $x_1$ axis and the $x_2$ axis is open, and is an open interval of $C$ if and only if its projection to both the $x_1$ axis and the $x_2$ axis is an open interval. Since every open set on the real line is the union of open intervals, it is not hard to see that every open piece of $C$ is the union of open intervals of $C$. Then, since every open interval of $C$ is a connected piece of $C$, our work above tells us that the claim holds if $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq \mathrm{LR}(p,q)$ are each open pieces of $C$.

Suppose, by way of contradiction, that for some $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq LR(p,q)$, $m_2(X)/m_1(X) > m_2(Y)/m_1(Y)$. Since $X$ and $Y$ are each measurable (with respect to both $m_1$ and $m_2$), we can approximate each of these fractions as closely as we wish by using open sets. It follows that there are open sets $X \subseteq \mathrm{UL}(p,q)$ and $Y \subseteq \mathrm{LR}(p,q)$ with $m_2(X)/m_1(X) > m_2(Y)/m_1(Y)$. This contradicts our conclusion in the previous paragraph and hence establishes the claim.

Returning to the proof of the theorem, we wish to show that $\mathrm{pr}_{12}\,\mathrm{pr}_{21} \leq 1$. Suppose, by way of contradiction, that $\mathrm{pr}_{12}\,\mathrm{pr}_{21} > 1$. Then, by the definition of $\mathrm{pr}_{12}$ and $\mathrm{pr}_{21}$, it follows that for some $X \subseteq \mathrm{UL}(p,q)$ and some $Y \subseteq \mathrm{LR}(p,q)$, $(m_2(X)/m_1(X))(m_1(Y)/m_2(Y)) > 1$. Then, $m_2(X)/m_1(X) > m_2(Y)/m_1(Y)$. This contradicts the claim and hence establishes the theorem. ∎

## 3. THE $n = 3$ CASE

The statement of Theorem 3 generalizes in a natural way to three players. Unfortunately, this generalization is false. In this section, we shall examine why this is so. This will involve the consideration of notions of symmetry appropriate for the $n = 3$ context. We will close with open questions about the possibility of generalizing Theorem 3.

Suppose that $C$ is our cake and that $m_1$, $m_2$, and $m_3$ are measures on $C$. What do we know about $\mathrm{IPS}(C; m_1, m_2, m_3)$? Certainly, the first four of the five conditions of Theorem 3 generalize to our present three-player context:

a. $\mathrm{IPS}(C; m_1, m_2, m_3) \subseteq [0,1] \times [0,1] \times [0,1]$;

b. $(1,0,0), (0,1,0), (0,0,1) \in \mathrm{IPS}(C; m_1, m_2, m_3)$;

c. $\mathrm{IPS}(C; m_1, m_2, m_3)$ is closed;

d. $\mathrm{IPS}(C; m_1, m_2, m_3)$ is convex.

Condition a follows from the fact that our measures take values in the interval $[0, 1]$. Condition b follows by considering what happens if all of $C$ is given to Player 1, then to Player 2, and then to Player 3. Conditions c and d follow from the theorem of Dvoretsky, Wald, and Wolfovitz. As discussed previously, conditions b and d tell us that the 2-simplex is in $\mathrm{IPS}(C; m_1, m_2, m_3)$. Also, $\mathrm{IPS}(C; m_1, m_2, m_3)$ consists precisely of the 2-simplex if and only if the three measures are identical.

What about condition e of Theorem 3, symmetry about the point $\langle \frac{1}{2}, \frac{1}{2} \rangle$? The obvious generalization of this condition to the three-player context is symmetry about $\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$. It turns out that, in general, this does not hold, and we shall give an example of this shortly. We first examine the sort of symmetry that does exist in this context.

In searching for symmetry in the $n = 3$ context, let us consider how to generalize the simple proof of symmetry about $\langle \frac{1}{2}, \frac{1}{2} \rangle$ in the $n = 2$ context. In that case, we found symmetry by starting with some partition and then having the two players exchange pieces. The following theorem and its proof can be viewed as a direct generalization of this idea.

THEOREM 4. *Suppose that $C$ is a cake, $m_1$, $m_2$, and $m_3$ are measures on $C$; and $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. Then there is a triangle $T$ such that*

a. *$(a, b, c)$ is a vertex of $T$;*

b. *$T$ and its interior are contained in* $\mathrm{IPS}(C; m_1, m_2, m_3)$;

c. *$T$ has centroid $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (i.e., $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the point of intersection of the line segments connecting each vertex of $T$ with the midpoint of the opposite side).*

*Proof.* We assume that $C$ is a cake, $m_1$, $m_2$, and $m_3$ are measures on $C$, and $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. Then, for some partition $\langle P_1, P_2, P_3 \rangle$ of $C$, $(m_1(P_1), m_2(P_2), m_3(P_3)) = (a, b, c)$. We obtain two new partitions by having each player "pass to the right" and then "pass to the right" again. More specifically, we consider the partitions $\langle P_3, P_1, P_2 \rangle$ and $\langle P_2, P_3, P_1 \rangle$.

What are the corresponding points in the IPS? Our starting point was $(m_1(P_1), m_2(P_2), m_3(P_3))$. Our two new points are $(m_1(P_3), m_2(P_1), m_3(P_2))$ and $(m_1(P_2), m_2(P_3), m_3(P_1))$. Let $T$ be the triangle formed by these three points. Then, recalling that $(m_1(P_1), m_2(P_2), m_3(P_3)) = (a, b, c)$, we know that $(a, b, c)$ is a vertex of $T$. Also, since all three vertices are in $\mathrm{IPS}(C; m_1, m_2, m_3)$, convexity implies that $T$ and its interior are contained in $\mathrm{IPS}(C; m_1, m_2, m_3)$. We must show that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the centroid of $T$.

Since $P_1$, $P_2$, and $P_3$ are pairwise disjoint and have union $C$, it follows that

$$\begin{aligned} m_1(P_1) + m_1(P_2) + m_1(P_3) &= m_1(P_1 \cup P_2 \cup P_3) \\ &= m_1(C) = 1. \end{aligned}$$

Similarly, $m_2(P_1) + m_2(P_2) + m_2(P_3) = 1$ and $m_3(P_1) + m_3(P_2) + m_3(P_3) = 1$. Thus $(m_1(P_1), m_2(P_2), m_3(P_3))$, $(m_1(P_3), m_2(P_1), m_3(P_2))$, and $(m_1(P_2), m_2(P_3), m_3(P_1))$ have coordinate sum $(1, 1, 1)$ and hence coordinate average $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is straightforward to show that this implies that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the centroid of the triangle formed by these three points, i.e., of $T$. ▮

The idea of a "triangle" in the theorem may be somewhat misleading. The triangle may be degenerate in the sense that two or possibly all three of the vertices may be identical. For example, given $C$, $m_1$, $m_2$, and $m_3$ as in the theorem, it follows from Lyapounov's theorem (see, for example, [6] or [8]) that there exists a partition of $C$ among the three players that all agree is a partition into three equal pieces. In this case, the three points of $\mathrm{IPS}(C; m_1, m_2, m_3)$ are all the same point, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. (Of course, given that the three points have centroid $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the three points must all equal $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ if they are equal to each other.)

The essential information on symmetry contained within the theorem is given to us by the following two corollaries.

Corollary 1. *Suppose that $C$ is a cake, $m_1$, $m_2$, and $m_3$ are measures on $C$, and $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. If $(p, q, r)$ is the point such that*

a. *$(a, b, c)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and $(p, q, r)$ are collinear, with $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ between $(a, b, c)$ and $(p, q, r)$ and*

b. *the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $(p, q, r)$ is half the distance from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ to $(a, b, c)$,*

*then $(p, q, r) \in \mathrm{IPS}(C; m_1, m_2, m_3)$.*

*Proof.* It is a standard fact of geometry that the distance from the centroid of a triangle to the midpoint of a side is half the distance from the centroid to the vertex opposite that side. Thus, for any point $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$, the point $(p, q, r)$ of the corollary is on a side of any triangle given by the theorem. Then, the theorem tells us that $(p, q, r) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. ▮

Corollary 2. *Suppose that $C$ is a cake, $m_1$, $m_2$, and $m_3$ are measures on $C$, and $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. Then $(\frac{1-a}{2}, \frac{1-b}{2}, \frac{1-c}{2}) \in \mathrm{IPS}(C; m_1, m_2, m_3)$.*

*Proof.* Suppose that $(a, b, c) \in \mathrm{IPS}(C; m_1, m_2, m_3)$ and $(p, q, r)$ is as in Corollary 1. Then

$$\begin{aligned}(p, q, r) &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3} - a, \frac{1}{3} - b, \frac{1}{3} - c\right) \\ &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{1 - 3a}{6}, \frac{1 - 3b}{6}, \frac{1 - 3c}{6}\right) \\ &= \left(\frac{1 - a}{2}, \frac{1 - b}{2}, \frac{1 - c}{2}\right).\end{aligned}$$

Hence, $(\frac{1-a}{2}, \frac{1-b}{2}, \frac{1-c}{2}) \in \mathrm{IPS}(C; m_1, m_2, m_3)$.

There is another approach to this proof. Starting with any partition $\langle P_1, P_2, P_3 \rangle$, it follows from Lyapounov's theorem that $P_1$ can be divided into two pieces in such a way that all three players believe that these two pieces are of equal size. Similarly, $P_2$ and $P_3$ can be so divided. Each player then gives one of the two pieces of his or her original piece to each of the other two players. It is straightforward to show that if the point in $\mathrm{IPS}(C; m_1, m_2, m_3)$ corresponding to our original partition is $(a, b, c)$, then the point corresponding to the new partition is $(\frac{1-a}{2}, \frac{1-b}{2}, \frac{1-c}{2})$. Hence $(\frac{1-a}{2}, \frac{1-b}{2}, \frac{1-c}{2}) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. ∎

How does Corollary 1 fit in with what would be the natural generalization of condition e of Theorem 3? That natural generalization to the $n = 3$ context would say that the IPS is centrally symmetric about the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Another way to state this is, given any point $(a, b, c) \in$ IPS, if we travel from that point to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and then continue along that same line precisely that same distance, then we arrive at another point of the IPS. (Of course, by convexity, all points of this segment would be in the IPS.) Our present result is weaker. It says that given any point $(a, b, c) \in$ IPS, if we travel from that point to the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and then continue along that same line for precisely half that distance, then we arrive at another point of the IPS.

Note that Corollary 1 does not imply that $\mathrm{IPS}(C; m_1, m_2, m_3)$ is not centrally symmetric about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The failure of symmetry about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is implied by the following.

THEOREM 5. *Corollary* 1 *is the best possible result of this sort. In other words, if the number* $\frac{1}{2}$ *in the corollary is replaced by any larger number, the statement would not be true.*

*Proof.* It suffices to present an example of a cake $C$; measures $m_1$, $m_2$, and $m_3$ on $C$; and a point $(a, b, c)$ on the outer boundary of $\mathrm{IPS}(C; m_1,$

$m_2, m_3$), such that if $(p, q, r)$ is related to $(a, b, c)$ as in Corollary 1, then $(p, q, r)$ is on the inner boundary of IPS($C; m_1, m_2, m_3$).

Let our cake $C$ be the interval $[0, 3)$ and let $m_{\text{Leb}}$ be Lebesgue measure on this interval. We need to define $m_1$, $m_2$, and $m_3$ on $C$. Let $X \subseteq C$ be arbitrary. We can write $X$ as $X_1 \cup X_2 \cup X_3$, where $X_1 \subseteq [0, 1)$, $X_2 \subseteq [1, 2)$, and $X_3 \subseteq [2, 3)$. Then, we define $m_1(X)$, $m_2(X)$, and $m_3(X)$ as follows:

$$m_1(X) = .6m_{\text{Leb}}(X_1) + .2m_{\text{Leb}}(X_2) + .2m_{\text{Leb}}(X_3)$$
$$m_2(X) = .2m_{\text{Leb}}(X_1) + .6m_{\text{Leb}}(X_2) + .2m_{\text{Leb}}(X_3)$$
$$m_3(X) = .2m_{\text{Leb}}(X_1) + .2m_{\text{Leb}}(X_2) + .6m_{\text{Leb}}(X_3).$$

It is easy to verify that $m_1$, $m_2$, and $m_3$ are (countably additive, non-atomic probability) measures on $C$.

Consider the partition $P = \langle [0, 1), [1, 2), [2, 3) \rangle$ of $C$. We claim that this partition is Pareto maximal. Let us compute the maximality partition ratios.

To compute $\text{pr}_{12}$, we note that for any $Y \subseteq [0, 1)$, $m_2(Y)/m_1(Y) = .2m_{\text{Leb}}(Y)/.6m_{\text{Leb}}(Y) = \frac{1}{3}$. Hence, $\text{pr}_{12} = \frac{1}{3}$. The other maximality partition ratios are computed similarly, and we find that all maximality partition ratios equal $\frac{1}{3}$.

It is then easy to check all possible maximality cyclic products:

$$\text{pr}_{12}\,\text{pr}_{21} = \text{pr}_{13}\,\text{pr}_{31} = \text{pr}_{23}\,\text{pr}_{32} = \left(\tfrac{1}{3}\right)^2 = \tfrac{1}{9}$$
$$\text{pr}_{12}\,\text{pr}_{23}\,\text{pr}_{31} = \text{pr}_{13}\,\text{pr}_{32}\,\text{pr}_{21} = \left(\tfrac{1}{3}\right)^3 = \tfrac{1}{27}.$$

Thus, we see that all maximality cyclic products are less than one. Hence, by Theorem 1, $P$ is a Pareto maximal partition of $C$. This tells us that the corresponding point of IPS($C; m_1, m_2, m_3$) is on the outer boundary. This point is

$$\begin{aligned}(m_1([0,1)), m_2([1,2)), m_3([2,3)))& \\ &= (.6m_{\text{Leb}}([0,1)), .6m_{\text{Leb}}([1,2)), .6m_{\text{Leb}}([2,3))) \\ &= (.6, .6, .6).\end{aligned}$$

We know that $(\frac{1-.6}{2}, \frac{1-.6}{2}, \frac{1-.6}{2}) = (.2, .2, .2)$ is related to $(.6, .6, .6)$ as $(p, q, r)$ is related to $(a, b, c)$ in Corollary 1, and this corollary tells us that $(.2, .2, .2) \in$ IPS($C; m_1, m_2, m_3$). To prove Theorem 5, it suffices to show that $(.2, .2, .2)$ is on the inner boundary of the IPS. To establish this, we show that $(.2, .2, .2)$ corresponds to a Pareto minimal partition.

How can we find a partition that corresponds to the point $(.2, .2, .2)$? The key lies in our second approach to the proof of Corollary 2. Each player's piece (according to partition $P$) can be divided into what each of the three players believes are two equal pieces. (In general, this step requires Lyapounov's theorem. However, in this case, Lyapounov's theorem is not needed, since all players will agree about what is a division of $[0, 1)$ or $[1, 2)$ or $[2, 3)$ into two equal pieces.) Each player then gives one of these two pieces to each of the other two players.

Proceeding with this idea, we consider the partition $Q = \langle [1.5, 2.5), [0, .5) \cup [2.5, 3), [.5, 1.5) \rangle$. It is straightforward to verify that this partition corresponds to the point $(.2, .2, .2) \in \mathrm{IPS}(C; m_1, m_2, m_3)$. We wish to show that $Q$ is Pareto minimal. We compute the minimality partition ratios.

To compute $\mathrm{qr}_{12}$, suppose that $Y \subseteq [1.5, 2.5)$. Then,

$$\frac{m_2(Y)}{m_1(Y)} = \frac{m_2(Y \cap [1.5, 2)) + m_2(Y \cap [2, 2.5))}{m_1(Y \cap [1.5, 2)) + m_1(Y \cap [2, 2.5))}$$
$$= \frac{.6 m_{\mathrm{Leb}}(Y \cap [1.5, 2)) + .2 m_{\mathrm{Leb}}(Y \cap [2, 2.5))}{.2 m_{\mathrm{Leb}}(Y \cap [1.5, 2)) + .2 m_{\mathrm{Leb}}(Y \cap [2, 2.5))}.$$

This tells us that

i. If $Y \subseteq [1.5, 2)$, then

$$\frac{m_2(Y)}{m_1(Y)} = \frac{.6 m_{\mathrm{Leb}}(Y \cap [1.5, 2))}{.2 m_{\mathrm{Leb}}(Y \cap [1.5, 2))} = 3.$$

ii. If $Y \subseteq [2, 2.5)$, then

$$\frac{m_2(Y)}{m_1(Y)} = \frac{.2 m_{\mathrm{Leb}}(Y \cap [1.5, 2))}{.2 m_{\mathrm{Leb}}(Y \cap [1.5, 2))} = 1.$$

iii. If $Y \not\subseteq [1.5, 2)$ and $Y \not\subseteq [2, 2.5)$, then $1 \leq m_2(Y)/m_1(Y) \leq 3$.

Thus, $\mathrm{qr}_{12} = \inf\{m_2(Y)/m_1(Y) : Y \subseteq [1.5, 2.5),\ m_1(Y) \neq 0\} = 1$. The other minimality partition ratios are computed similarly, and we find that all minimality partition ratios equal 1. This implies that all minimality cyclic products are equal to 1, and so, by Theorem 2, $Q$ is a Pareto minimal partition of $C$. It follows that $(.2, .2, .2)$ is on the inner boundary of $\mathrm{IPS}(C; m_1, m_2, m_3)$. This establishes the theorem. ∎

Theorem 5 tells us that, in general, we do not have the symmetry about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ that we may have expected, given the symmetry about $(\frac{1}{2}, \frac{1}{2})$ that held in the $n = 2$ context. However, Corollaries 1 and 2 do tell us that the outer boundary of the IPS and the inner boundary of the IPS have some connection. We cannot, for example, have the outer boundary far from the simplex and have the inner boundary very close to the simplex. Can we say more? In other words:

OPEN QUESTION. *Is there anything specific we can say about any connection between the shape of the outer boundary and the inner boundary of the IPS for $n = 3$?*

It appears that a generalization of Theorem 3 to the $n = 3$ context would necessitate a clear affirmative answer to this question. In the absence of such an answer, a natural move is to disconnect the issues of the shape of the outer boundary and the inner boundary. In other words, we can ask, what are the possible shapes of the "outer IPS," i.e., the subset of the IPS consisting of all points of the IPS that are not on the same side of the simplex as the origin? Clearly, this is the same as asking, what are the possible shapes of the outer boundary of the IPS?

The same thing can be asked about the "inner IPS" or the inner boundary of the IPS. Then, one might ask whether the answer to each of these questions is provided by something like Theorem 3, with only the obvious variation of condition a ($A \subseteq \{(p,q,r) \in [0,1] \times [0,1] \times [0,1] : p + q + r \geq 1\}$) and conditions b, c, and d. Such a theorem is certainly true for the separate "outer" and "inner" questions for $n = 2$. The following observation establishes that this is not so in the $n = 3$ context.

*Observation.* Pick any point $(p,q,r)$ with $0 \leq p \leq 1$, $0 \leq q \leq 1$, $0 \leq r \leq 1$, and $p + q + r > 1$. If the "outer boundary" version of Theorem 3 for the $n = 3$ context were true, then there would exist a cake and corresponding measures such that the associated outer IPS is the convex hull of $\{(1,0,0),(0,1,0),(0,0,1),(p,q,r)\}$. Equivalently, there would exist such a cake and corresponding measures so that the outer boundary of the associated IPS consists of the three triangles determined by the point $(a,b,c)$ and each of the three pairs of points from $\{(1,0,0),(0,1,0),(0,0,1)\}$. We claim that this is not possible. Let $A$ denote this outer IPS and let $A^{\text{out}}$ denote its outer boundary.

Suppose, by way of contradiction, that cake $C$ and measures $m_1$, $m_2$, and $m_3$ on $C$ are such that the outer boundary of $\text{IPS}(C; m_1, m_2, m_3)$ is $A^{\text{out}}$. The fact that $\text{IPS}(C; m_1, m_2, m_3)$ consists of more that just the simplex tells us that the measures $m_1$, $m_2$, and $m_3$ are not identical. Suppose, without loss of generality, that $m_1 \neq m_2$, and consider the possible Pareto maximal partitions of $C$ in which Player 3 gets no cake.

The corresponding subset of the outer boundary of the IPS is a curve in the $x_3 = 0$ plane. Since $m_1 \neq m_2$, this curve is not the straight-line segment from $(1, 0, 0)$ to $(0, 1, 0)$. This situation is illustrated in Fig. 2. The three triangles making up $A^{\text{out}}$ are shaded. The curve on the outer boundary corresponding to Pareto maximal partitions in which Player 3 gets no cake is shown. This curve is clearly not a subset of $A^{\text{out}}$, since $A^{\text{out}}$'s intersection with the $x_3 = 0$ plane consists of the line segment from $(1, 0, 0)$ to $(0, 1, 0)$.

We close by asking whether Theorem 3 can be generalized to the $n = 3$ context if we focus separately on the outer and inner IPS, drop the symmetry condition, and add a new condition to require that the appropriate curve in the $x_3 = 0$ plane, and the analogous curves in the $x_1 = 0$ and the $x_2 = 0$ planes, be included whenever a point $(p, q, r)$, as in the observation above, is included. More specifically, we ask the following:

OPEN QUESTION. *Is there a function f with domain the unit cube and whose values are subsets of* $\mathbb{R}^3$, *so that the following result is true: Let A be a subset of* $\mathbb{R}^3$. *There exists a cake C and measures* $m_1$, $m_2$, *and* $m_3$ *on C such that A is the "outer* IPS$(C; m_1, m_2, m_3)$" (*i.e.*, $A = \{(p, q, r) \in \text{IPS}(C; m_1, m_2, m_3) : p + q + r \geq 1\}$) *if and only if A satisfies the following*

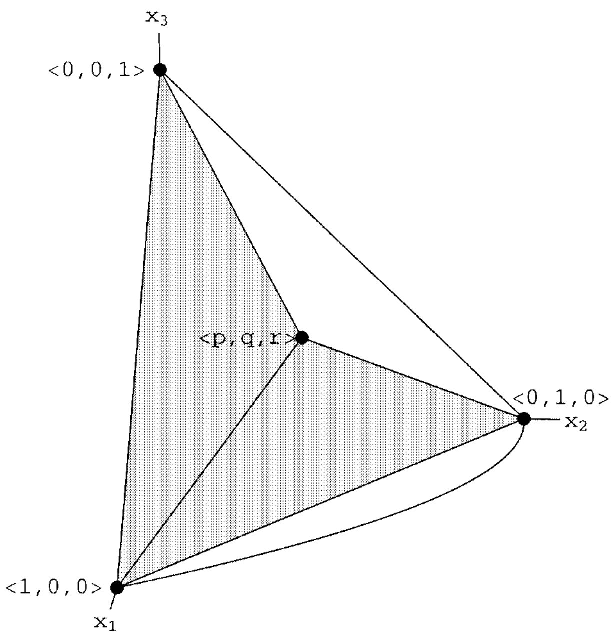


FIGURE 2

*five conditions*:

i. $A \subseteq \{(p, q, r) \in [0,1] \times [0,1] \times [0,1] : p + q + r \geq 1\}$;

ii. $(1,0,0), (0,1,0), (0,0,1) \in A$;

iii. $A$ *is closed*;

iv. $A$ *is convex*;

v. $A$ *is closed under* $f$ (*i.e.*, *if* $(p, q, r) \in A$, *then* $f(p, q, r) \subseteq A$).

The intuition here is that $f$ is producing the necessary three curves in the three coordinate planes, as discussed above.

A natural next question is to pursue an analogous result for dimensions $n > 3$.

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